

2.39. Indirect Deduction Strategy

We promised that Indirect Deduction (ID) could establish the validity of arguments that the deductive rules alone can't – offering the following as a simple example.

$$\frac{1. \sim P}{\therefore \sim(P \wedge Q)}$$

With ID in hand it's easy to establish the validity of this argument.

1.	$\sim P$	
2.	$\sim\sim(P \wedge Q)$	Get $\sim(P \wedge Q)$ (ID) AID
3.	$(P \wedge Q)$	2, $\sim\sim$
4.	P	3, $\wedge\sim$
5.	$\sim P$	1, R
6.	$\sim(P \wedge Q)$	2, 4, 5, ID

And it's simple enough to make good on an earlier claim: that any argument deducible without using ID can also be deduced via indirect deduction. For we can always wrap an ID box around a completed deduction.

The following boring little deduction, for instance, reaches the conclusion, “Q,” without appeal to Indirect Deduction.

1. $(P \vee Q)$ 2. $\sim P$ <hr style="border: 1px solid black; margin: 5px 0;"/> 3. Q	Get: Q 1, 2, \vee -
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But all these steps can be preserved in an Indirect Deduction – simply adding an ID box, and its accompanying AID.¹

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|----|--------------|-------------------|
| 1. | $(P \vee Q)$ | |
| 2. | $\sim P$ | |
| 3. | $\sim Q$ | Get Q (ID)
AID |
| 4. | Q | 1, 2, \vee - |
| 5. | Q | 3, 4, ID |

So we lose no deductive power by pursuing every deduction through ID. And since (as in our earlier example) some arguments can be deduced *only* through ID, our strategy will be to use ID for each deduction – unless there’s some obvious, simple way of reaching the conclusion through the rules alone.

Indirect Deduction Strategy: automatically use ID (unless there’s an obvious way of reaching the conclusion without ID).

Now it’s already a feature of our deductive system that if an argument in the formal language is valid, ID and the seven rules can demonstrate this – deducing, within the ID box, some sentence and its negation. But from those two sentences we can deduce any further sentences we please. For as noted elsewhere, an inconsistent set of sentences entails any (and every) sentence.² (Indeed, this explosive entailment is what makes inconsistency so unacceptable.)

That means that whenever a sentence and its negation can be deduced, **every** sentence and its negation can be.

¹ The justification of Line 5 cites only two line numbers (3 and 4) rather than the usual three, because here “ $\sim Q$ ” does double duty: as the AID, and also as one of the two mutually contradictory sentences.

² In 2.16.

In that last deduction, for instance, we didn't need to rest content with "Q" and " $\sim Q$ ". We could get whatever pair of sentences we wanted.

1.	$(P \vee Q)$	
2.	$\sim P$	
		Get Q (ID)
3.	$\sim Q$	AID
4.	Q	1, 2, $\vee-$
5.	$(Q \vee X)$	4, $\vee+$
6.	X	3, 5, $\vee-$
7.	$(Q \vee \sim X)$	4, $\vee+$
8.	$\sim X$	3, 7, $\vee-$
9.	Q	3, 6, 8, ID

So in an ID we always have a choice of which pair of sentences to use. But it's just unnecessary work to choose a pair, such as $\{X, \sim X\}$, built from a sentence letter appearing nowhere in the premises. In general, it saves time and labor to choose sentences which use a sentence letter appearing already in the argument.

Indirect Deduction Strategy: choose a pair of sentences that feature a sentence letter already appearing in previous lines.

We can extend that point further: not only should our pair of sentences feature a sentence letter appearing earlier in the deduction, but when possible we should **use a sentence we already have** as one member of this pair. For then we only need to come up with the other half – effectively cutting the ID work in half.

Indirect Deduction Strategy: try to use a sentence you already have as half of the contradictory pair of sentences, and then get the other half.

The following argument is a simple example (though one deducible only through ID).³

1. We're having truffles.	1. P
2. We're not having <i>both</i> truffles <i>and</i> grog.	2. $\sim(P \wedge Q)$
<hr/>	
\therefore We're not having grog.	$\therefore \sim Q$

Once we've assumed the negation of the conclusion, we have one application of $\sim\sim$ before running dry on Elim rules. In this situation our strategy is to use the Intro rules to set up a sentence – either the missing ingredient for an Elim rule, or the very sentence on the “Get” line.

1. P	
2. $\sim(P \wedge Q)$	
3. $\sim\sim Q$	Get $\sim Q$ (ID) AID
4. Q	3, $\sim\sim$

But setting up an Elim rule is no good, since there's no conjunction, disjunction, or double-negation to break down.

And before attempting to build the sentence on the “Get” line, we need to keep in mind that in an ID we reach that sentence indirectly – by *first* assembling a contradictory pair of sentences within the ID box. We can't close that box, and reach the sentence on the “Get” line, until we have such a pair of sentences.

³ This form of argument was recognized a valid by the ancient Stoic logicians, who treated it as one of their basic inference patterns – in effect, a basic rule of deduction. See (Mates XX: XX).

Setting out in search of a sentence and its negation, our latest bit of strategy comes to bear: instead of building up both sentences, we try to use a sentence we already have as half of the desired pair. We then only need to build the other half.

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|----|--------------------|-------------------|
| 1. | P | |
| 2. | $\sim(P \wedge Q)$ | |
| | | Get $\sim Q$ (ID) |
| 3. | $\sim\sim Q$ | AID |
| 4. | Q | 3, $\sim\sim$ |

Both “Q” and “ $\sim\sim Q$ ” call for “ $\sim Q$ ” as the other half of the contradictory pair; but no Intro rule builds a single negation. That likewise rules out “P,” which takes “ $\sim P$ ” as the other half of the contradictory pair.

But “ $\sim(P \wedge Q)$ ” is more promising, since the other half of the pair is then the conjunction “ $(P \wedge Q)$ ” – and there is an Intro rule for building a conjunction.

Applying $\wedge+$ to Lines 1 and 4 (then bringing down Line 2 through R) completes the ID.

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|----|--------------------|-------------------|
| 1. | P | |
| 2. | $\sim(P \wedge Q)$ | |
| | | Get $\sim Q$ (ID) |
| 3. | $\sim\sim Q$ | AID |
| 4. | Q | 3, $\sim\sim$ |
| 5. | $(P \wedge Q)$ | 1, 4, $\wedge+$ |
| 6. | $\sim(P \wedge Q)$ | 2, R |
| 7. | $\sim Q$ | 3, 5, 6, ID |

Note that in constructing the missing half of the contradictory pair of sentences, we’ve just found a new application for a familiar bit of strategy: using the Intro rules to break a deductive logjam. For within an ID, finding such a pair of sentences clearly advances the deduction.

And ID itself offers a surprising further way of reaching a sentence which breaks such a logjam. The following little (valid) argument illustrates how.

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|--|--|
| 1. Either we're having both truffles and grog,
or we're having truffles without having grog.
<hr style="width: 50%; margin-left: 0;"/> \therefore We're having truffles. | 1. $((P \wedge Q) \vee (P \wedge \sim Q))$
<hr style="width: 50%; margin-left: 0;"/> $\therefore P$ |
|--|--|

Once past the AID, the deduction hits a wall, as there are no opportunities to apply any Elim rule.

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|---|--|
| 1. $((P \wedge Q) \vee (P \wedge \sim Q))$
2. $\left \begin{array}{l} \sim P \end{array} \right.$ | <hr style="width: 50%; margin-left: 0;"/> Get: P (ID)
(AID) |
|---|--|

Our default strategy in such a case is to ‘set up’ a sentence which gets the deduction moving again. But here using a sentence we already have, as half of a contradictory pair, doesn’t look like an option. For both Lines 1 and 2, we lack the ingredients to construct the opposing sentence – “ $\sim((P \wedge Q) \vee (P \wedge \sim Q))$ ” and “ $\sim\sim P$,” respectively.

Setting up an Elim rule looks more promising, since Line 1 is a disjunction. To apply \vee - we then need either “ $\sim(P \wedge Q)$ ” or “ $\sim(P \wedge \sim Q)$ ”. But since neither is the sort of sentence an Intro rule could build, the chances of advancing the deduction again seem dim.

Yet all hope is not lost. Consider: if we had Lines 1 and 2 as premises, and wanted to deduce “ $\sim(P \wedge Q)$ ” or “ $\sim(P \wedge \sim Q)$ ” from them, we’d know exactly what to do: write a “Get” line for that desired sentence, and start an ID.

Why not do the same here? True, we’ve so far only written a “Get” line immediately after the premises of a deduction. But there’s no official restriction on where one can be introduced. So we declare this further un-jamming strategy: **if no other moves are open in a deduction, write a “Get” line for the desired sentence.**

Indirect Deduction Strategy: if the deduction needs a particular sentence to advance, and no application of the rules can obtain that sentence, write a “Get” line for that sentence.

(Of course we use ID on that “Get” line. For otherwise we’d have only the rules and the existing lines available to us; and that much left us at a deductive standstill.)

We also keep in mind a point stressed in the last example: that while our ultimate goal in an ID is of course the sentence on the “Get” line, we reach it only by first deducing, within the ID box, some sentence and its negation. So though in our current ID we are indeed trying to get the conclusion “P,” we’ll do so only by first getting such a pair of sentences.

For that reason, it would be a waste of time to write a new “Get” line for “P”.

💀 A Fruitless Maneuver 💀

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|----|---|--|
| 1. | $((P \wedge Q) \vee (P \wedge \sim Q))$ | |
| | <div style="border-bottom: 1px solid black; margin-bottom: 10px;"> <div style="display: flex; justify-content: space-between;"> <div style="border-left: 1px solid black; padding-left: 10px;">2.</div> <div>$\sim P$</div> <div style="flex-grow: 1;"></div> <div>Get: P (ID)</div> </div> <div style="display: flex; justify-content: space-between;"> <div style="border-left: 1px solid black; padding-left: 10px;">3.</div> <div>$\sim P$</div> <div style="flex-grow: 1;"></div> <div>AID</div> </div> </div> <div style="display: flex; justify-content: space-between;"> <div style="border-left: 1px solid black; padding-left: 10px;">2.</div> <div>$\sim P$</div> <div style="flex-grow: 1;"></div> <div>(AID)</div> </div> | |

The second ID leaves us searching for a contradictory pair of sentences all over again, **armed with no new sentences** (since the new AID on Line 3 is just Line 2 again).

To avoid such an infinite loop, the smaller ID should instead have as AID **a sentence not already had**. That means: there must be a new sentence on the second “Get” line.

As in deductions generally, if we’re stuck we try to set up an Elim rule – specifically, \vee –. Here, we choose the negation of the left side of Line 1.⁴

1.	$((P \wedge Q) \vee (P \wedge \sim Q))$	
		Get: P (ID)
2.	$\sim P$	(AID)
		Get: $\sim(P \wedge Q)$ (ID)
3.	$\sim\sim(P \wedge Q)$	AID

Automatically applying two Elim rules, \sim – and \wedge –, yields a sentence and its negation: “ $\sim P$ ” on Line 2, and “ P ” on Line 5. Repeating down Line 2 leaves a contradictory pair of sentences within the smaller ID box.

1.	$((P \wedge Q) \vee (P \wedge \sim Q))$	
		Get: P (ID)
2.	$\sim P$	(AID)
		Get: $\sim(P \wedge Q)$ (ID)
3.	$\sim\sim(P \wedge Q)$	AID
4.	$(P \wedge Q)$	3, \sim –
5.	P	8, \wedge –
6.	$\sim P$	2, R
7.	$\sim(P \wedge Q)$	3, 5, 6, ID

With the inner ID box closed, our deductive logjam is at last freed up.

“ $\sim(P \wedge Q)$ ” on Line 7 is the missing ingredient for inflicting \vee – on Line 1, yielding “ $(P \wedge \sim Q)$ ”.

⁴ Though the negation of the right half would work just as well, and be just as easy to deduce.

Automatic application of $\wedge-$ to Line 8 yields “P”. And with both “ $\sim P$ ” (Line 2) and “P” (Line 9) in this larger box, our ID is complete.

1.	$((P \wedge Q) \vee (P \wedge \sim Q))$	
		Get: P (ID)
2.	$\sim P$	(AID)
		Get: $\sim(P \wedge Q)$ (ID)
3.	$\sim\sim(P \wedge Q)$	AID
4.	$(P \wedge Q)$	3, $\sim-$
5.	P	4, $\wedge-$
6.	$\sim P$	2, R
7.	$\sim(P \wedge Q)$	3, 5, 6, ID
8.	$(P \wedge \sim Q)$	1, 7, $\vee-$
9.	P	8, $\wedge-$
10.	P	2, 9, ID

Such ‘embedded IDs’ – repeating the ID-building procedure within a larger ID – thus rank alongside the Intro rules as a tool for ‘setting up’ the sentence needed to complete a deduction or apply an Elim rule.

What we’re doing here, of course, is using Indirect Deduction **recursively**. Just as recursive construction rules embedded, e.g., a smaller negation within a larger one, so here we build a smaller ID as part of a larger ID. And just as we could cycle on the construction rules any finite number of times – yielding larger and larger formal sentences – so in what follows we allow embedding of any (finite) number of IDs within IDs, yielding IDs of arbitrarily large proportions.

With embedded IDs added to our bag of tricks, we have in hand all the deductive tools needed to demonstrate the validity of any valid argument in the formal language. All we add from here are new applications of these deductive tools, and strategies for streamlining their use.

Summary: Indirect Deduction Strategy

- Automatically use ID for each deduction (unless there is an obvious way of reaching the conclusion without ID).
- As the needed contradictory pair of sentences – some sentence, and its negation – try to choose sentences featuring a sentence letter that already appears on some previous line.
- As the needed contradictory pair of sentences, try to use a sentence you already have as one of those two sentences, then deduce the other half.
- If a sentence is needed to advance the deduction and can't be reached by applying the deductive rules to existing lines, write a “Get” line for that sentence and use ID for the new “Get” line. (Here ID acts on a par with the Intro rules to ‘set up’ a needed sentence.)